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Generalized Bernstein polynomials with Pollaczek weight

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Abstract

Bernstein polynomials are a useful tool for approximating functions. In this paper, we extend the applicability of this operator to a certain class of locally continuous functions. To do so, we consider the Pollaczek weight

$$w(x) := \exp\left(-\frac{1}{\sqrt{x(1-x)}}\right), \quad 0 < x < 1,$$

which is rapidly decaying at the endpoints of the interval considered. In order to establish convergence theorems and error estimates, we need to introduce corresponding moduli of smoothness and K-functionals. Because of the unusual nature of this weight, we have to overcome a number of technical difficulties, but the equivalence of the moduli and K-functionals is a benefit interesting in itself. Similar investigations have been made in [B. Della Vecchia, G. Mastroianni, J. Szabados, Weighted approximation of functions with endpoint or inner singularities by Bernstein operators, Acta Math. Hungar. 103 (2004) 19–41] in connection with Jacobi weights.

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1. Introduction

Investigations concerning Bernstein polynomials frequently quote the fundamental monograph [5] of G.G. Lorentz. In this paper we will also make occasional references to estimates, relations presented in this book. We will be concerned with the Pollaczek weight

$$w(x) := \exp(-1/\varphi(x)), \quad \varphi(x) = \sqrt{x(1-x)}, 0 < x < 1.$$

Note that this weight does not satisfy the Szegő condition

$$\int_0^1 \frac{\log w(x)}{\sqrt{x(1-x)}} \mathrm{d}x > -\infty$$

which makes its role in approximation theoretical problems even more interesting. We will be dealing with the class of functions

$$C_w = \{f \in C^0(0,1) : \lim_{x(1-x)\to 0} (fw)(x) = 0\}$$

equipped with the norm

$$\|f\|_{C_w} = \sup_{0\le x\le 1} |(fw)(x)| = \|fw\|.$$

This means that we allow locally continuous functions in $(0, 1)$ which may go to infinity exponentially. Of course, ordinary Bernstein polynomials make no sense for these functions. Therefore we have to omit function values at the endpoints of the interval. Let

$$x_0 := \frac{3}{\sqrt{n}}, \qquad x_1 := \frac{3}{\sqrt{n}} + \frac{1}{n^{3/4}}, \qquad x_2 := 1 - \frac{3}{\sqrt{n}} - \frac{1}{n^{3/4}}, \qquad x_3 := 1 - \frac{3}{\sqrt{n}}, \tag{1.1}$$

and consider the linear functions

$$P_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1), \tag{1.2}$$

$$P_2(x) = \frac{x-x_3}{x_2-x_3} f(x_2) + \frac{x-x_2}{x_3-x_2} f(x_3) \tag{1.3}$$

interpolating $f \in C_w$ at x_0, x_1 and x_2, x_3, respectively. Further let $\psi(x) \in C^2(\mathbf{R})$ be such that $\psi(x) \equiv 0$ if $-\infty < x \le 0$ and $\psi(x) \equiv 1$ if $1 \le x < \infty$. With the notation

$$\psi_i(x) = \psi\left(\frac{x-x_i}{x_{i+1}-x_i}\right), \quad i = 0, 2, \tag{1.4}$$

we introduce the function

$$\begin{aligned} F_n(x) &= F_n(f,x) = (1-\psi_0(x))P_1(x) + \psi_0(x)(1-\psi_2(x))f(x) + \psi_2(x)P_2(x) \\ &= \begin{cases} P_1(x), & \text{if } 0 \le x \le x_0, \\ (1-\psi_0(x))P_1(x) + \psi_0(x)f(x), & \text{if } x_0 < x \le x_1, \\ f(x), & \text{if } x_1 < x < x_2, \\ (1-\psi_2(x))f(x) + \psi_2(x)P_2(x), & \text{if } x_2 < x < x_3, \\ P_2(x), & \text{if } x_3 \le x \le 1. \end{cases} \end{aligned} \tag{1.5}$$

Evidently, $F_n(f) \equiv f$ if f is a linear function. Finally, we put

$$B_n^\star(f,x) = B_n(F_n(f),x)$$

where

$$B_n(F_n, x) = \sum_{k=0}^{n} F_n\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

is the ordinary Bernstein operator. We introduce the Sobolev type space (see [2], Section 2, for a similar definition)

$$W^2 = \{f \in C_w : f' \in \mathrm{AC}(0,1) \text{ and } \|f''\varphi^2 w\| < \infty\},$$

and, for "t small", the K-functional is defined as follows:

$$K(f, t^2)_w = \inf_{g \in W^2} \{\|(f-g)w\| + t^2 \|g''\varphi^2 w\|\}.$$

Its main part is

$$\tilde{K}(C, f, t^2)_w = \sup_{0<h\le t} \inf_{g \in W^2} \{\|(f-g)w\|_{I_{Ch}} + h^2 \|g''\varphi^2 w\|_{I_{Ch}}\}$$

where $I(Ch) = [2Ch, 1-2Ch]$, $C \ge 1$. By definition, $\tilde{K}$ depends on the constant C and the following proposition holds:

Proposition 1. *If C and B are two constants ≥ 1, then*

$$\tilde{K}(C, f, t^2)_w \sim \tilde{K}(B, f, t^2)_w$$

where the constants in "$\sim$" are independent of f and t.

Now denote the main part of the K-functional as

$$\tilde{K}(f, t^2)_w := \tilde{K}(1, f, t^2)_w,$$

and define the modulus of continuity

$$\Omega_\varphi^2(C, f, t)_w = \sup_{0<h<t} \|w \Delta_{h\varphi}^2 f\|_{I_{Ch}},$$

where $C \ge 1$, and

$$\Delta_{h\varphi}^2 f(x) = f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x)).$$

The following proposition holds true.

Proposition 2. *For all $C \ge 1$ we have*

$$\tilde{K}(C, f, t)_w \sim \Omega_\varphi^2(C, f, t)_w$$

as $t \to 0$, and the constant in "$\sim$" is independent of f and t.

From Propositions 1 and 2 we can also deduce

$$\Omega_\varphi^2(B, f, t)_w \sim \Omega_\varphi^2(C, f, t)_w, \quad B, C \ge 1,$$

and, for simplicity, in the sequel we will write $\Omega_\varphi^2(f, t)_w$ for this modulus.

Then the complete modulus of smoothness of order 2 is defined as

$$\omega_\varphi^2(f, t)_w = \Omega_\varphi^2(f, t)_w + \inf_{q \in \mathcal{P}_1} \|(f-q)w\|_{C[0,2t]} + \inf_{q \in \mathcal{P}_1} \|(f-q)w\|_{C[1-2t,1]} \tag{1.6}$$

where, in general, $\mathcal{P}_n$ is the set of polynomials of degree at most n. The behavior of $\omega_\varphi^2(f,t)_w$ is independent of the possible constants in Ω_φ^2.

Propositions 1 and 2 will be proved in Section 5.

2. Results

In what follows $c > 0$ will always denote absolute constants, not necessarily the same at each occurrence.

Theorem 1. *We have*

$$\|(B_n^\star(f))w\| \le c\|fw\|, \quad f \in C_w, \tag{2.1}$$

$$\|(f - B_n^\star(f))w\| \le \frac{c}{n}\|f''\varphi^2 w\|, \quad f \in W_2, \tag{2.2}$$

$$\|(f - B_n^\star(f))w\| \le \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_w, \quad f \in C_w. \tag{2.3}$$

The next result shows the sharpness of (2.3) for Lipschitz classes.

Theorem 2. *We have*

$$\|(f - B_n^\star(f))w\| = O(n^{-\alpha/2}) \Leftrightarrow \omega_\varphi^2(f,h)_w = O(h^\alpha), \quad 0 < \alpha < 2.$$

Remark. We could have generalized the above results for the "nonsymmetric Pollaczek weights"

$$w(x) = \exp\left(-\frac{1}{x^a(1-x)^b}\right), \quad 0 < a, b < 1,$$

by proper modification of the parameters figuring in the proofs. However, this would have resulted in extremely complicated formulas while not giving any theoretical novelties. To illustrate the difficulties in this more general case, we mention that (1.1) should be replaced by

$$x_0 := \frac{3}{n^{\frac{1}{1+2a}}}, \qquad x_1 := \frac{3}{n^{\frac{1}{1+2a}}} + \frac{c_1}{n^{\frac{1+a}{1+2a}}}, \qquad x_2 := 1 - \frac{3}{n^{\frac{1}{1+2b}}} - \frac{c_2}{n^{\frac{1+b}{1+2b}}},$$
$$x_3 := 1 - \frac{3}{n^{\frac{1}{1+2b}}},$$

where $c_1, c_2 > 0$ are some constants.

3. Proof of Theorem 1, Part (2.1)

Lemma 1. *We have*

$$\frac{w(x)}{w(y)} \le \begin{cases} c\exp\left(-\dfrac{y-x}{4y\sqrt{x}}\right), & \text{if } 0 < x < y \le \dfrac{3}{4}, \\ \exp\left(2\dfrac{x-y}{x\sqrt{y}}\right) & \text{if } 0 < y < x \le \dfrac{3}{4}. \end{cases} \tag{3.1}$$

Of course, similar inequalities hold for the cases $\frac{1}{4} \le y, x < 1$.

Proof. For $0 < x < y \le 3/4$ we have

$$\frac{1}{\varphi(y)} - \frac{1}{\varphi(x)} = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{xy(1-y)}} + \frac{\sqrt{1-x}-\sqrt{1-y}}{\sqrt{x(1-x)(1-y)}}$$
$$\le -\frac{y-x}{2y\sqrt{x(1-y)}} + \frac{y-x}{(1-x)\sqrt{x(1-y)}} = -\frac{(y-x)(1-x-2y)}{2y(1-x)\sqrt{x(1-y)}}$$
$$\le -\frac{y-x}{4y\sqrt{x}}$$

provided that $x+2y \le \frac{1}{2}$. Now if $x+2y > \frac{1}{2}$ then we may assume that $0 < x \le \frac{1}{3}$ since otherwise the first statement in (3.1) holds trivially. Hence $\frac{1}{12} \le y \le \frac{3}{4}$ which implies $w(y) \ge c$, and thus

$$\frac{w(x)}{w(y)} \le c\exp\left(-\frac{1}{\sqrt{x(1-x)}}\right) \le c\mathrm{e}^{-1/\sqrt{x}} \le c\exp\left(-\frac{y-x}{4y\sqrt{x}}\right).$$

This proves the first inequality in (3.1). The second can be proved similarly; we omit the details. □

We now turn to the proof of (2.1). By symmetry, we may assume that $0 \le x \le 1/2$. We have by (1.1)–(1.3),

$$P_1(x) = f(x_0)(3n^{1/4}+1-n^{3/4}x) + f(x_1)(n^{3/4}x - 3n^{1/4}) \tag{3.2}$$

and

$$P_2(x) = f(x_3)(n^{3/4}x+3n^{1/4}+1-n^{3/4}) + f(x_2)(n^{3/4}-3n^{1/4}-n^{3/4}x). \tag{3.3}$$

Hence

$$B_n^\star(f,x) = B_n(F_n,x) = \sum_{0\le k\le 3\sqrt{n}} P_1\left(\frac{k}{n}\right) p_{n,k}(x)$$
$$+ \sum_{3\sqrt{n}<k\le 3\sqrt{n}+n^{1/4}} \left[\left(1-\psi_0\left(\frac{k}{n}\right)\right) P_1\left(\frac{k}{n}\right) + \psi_0\left(\frac{k}{n}\right) f\left(\frac{k}{n}\right)\right] p_{n,k}(x)$$
$$+ \sum_{3\sqrt{n}+n^{1/4}<k\le n-3\sqrt{n}-n^{1/4}} f\left(\frac{k}{n}\right) p_{n,k}(x)$$
$$+ \sum_{n-3\sqrt{n}-n^{1/4}<k\le n-3\sqrt{n}} \left[\left(1-\psi_3\left(\frac{k}{n}\right)\right) f\left(\frac{k}{n}\right) + \psi_3\left(\frac{k}{n}\right) P_2\left(\frac{k}{n}\right)\right] p_{n,k}(x)$$
$$+ \sum_{n-3\sqrt{n}<k\le n} P_2\left(\frac{k}{n}\right) p_{n,k}(x).$$

Thus, using the boundedness of the psi-functions (1.4),

$$|B_n^\star(f,x)| \le c \sum_{0\le k\le 3\sqrt{n}+n^{1/4}} \left|P_1\left(\frac{k}{n}\right)\right| p_{n,k}(x) + c \sum_{3\sqrt{n}<k\le n-3\sqrt{n}} \left|f\left(\frac{k}{n}\right)\right| p_{n,k}(x)$$
$$+ c \sum_{n-3\sqrt{n}-n^{1/4}<k\le n} \left|P_2\left(\frac{k}{n}\right)\right| p_{n,k}(x),$$

whence by

$$w(x_1) \sim w(x_0), \qquad w(x_2) \sim w(x_3)$$

(see Lemma 1) we obtain

$$\left|P_1\left(\frac{k}{n}\right)\right| \le \frac{c\|fw\|}{w(x_0)}\left(3n^{1/4}+1-\frac{k}{n^{1/4}}\right), \qquad \left|P_2\left(\frac{k}{n}\right)\right| \le \frac{c\|fw\|n^{3/4}}{w(x_3)}.$$

Thus

$$\begin{aligned} w(x)|B_n^{\star}(f,x)| \le c\|wf\| \Bigg[& \frac{w(x)}{n^{1/4}w(x_0)} \sum_{0\le k\le 3\sqrt{n}+n^{1/4}} (3\sqrt{n}+n^{1/4}-k)p_{n,k}(x) \\ & + \left(\sum_{3\sqrt{n}<k\le nx(1-\sqrt{x})} + \sum_{nx(1-\sqrt{x})<k\le 3n/4}\right)\frac{w(x)}{w(k/n)}p_{n,k}(x) \\ & + \frac{n^{3/4}}{w(x_3)}\sum_{3n/4<k\le n} p_{n,k}(x)\Bigg] := c\|wf\|\sum_{i=1}^{4} A_i(x). \end{aligned}$$

Estimate of $A_1(x)$. Case 1: $0 \le x \le x_0 + \frac{10}{n^{3/4}}$. Using the inequalities in Lemma 1 and

$$p_{n,k}(x) \le \frac{c}{\sqrt{nx}}\exp\left[-\frac{(nx-k)^2}{2nx}\right], \quad 0 \le x \le 1/2,\ |k-nx| \le n^{2/3} \tag{3.4}$$

(cf. Lorentz [5], Theorem 1.5.2), we obtain[1]

$$\begin{aligned} A_1(x) \le & \frac{1}{n^{3/4}\sqrt{x}}\exp\left(c\frac{x\sqrt{n}-3}{\sqrt{x}}\right)\sum_{0\le k\le 3\sqrt{n}+n^{1/4}}[\sqrt{n}(3-\sqrt{n}x)+n^{1/4}+(nx-k)] \\ & \times \exp\left[-\frac{(k-nx)^2}{2nx}\right] := \sum_{i=1}^{3} B_i(x). \end{aligned}$$

Now

$$\begin{aligned} B_1(x) & \le \frac{c}{n^{1/4}}\left\{\frac{|x\sqrt{n}-3|}{\sqrt{x}}\exp\left(c\frac{x\sqrt{n}-3}{\sqrt{x}}\right)\right\}\sum_{k=0}^{\infty}\exp\left[-\frac{(k-nx)^2}{2nx}\right] \\ & \le \frac{c}{n^{1/4}}\sum_{j=0}^{\infty}\exp\left(-\frac{j^2}{2nx}\right) \le c\sqrt{x}n^{1/4}\int_0^{\infty}\exp(-y^2)\mathrm{d}y \le c, \end{aligned}$$

since the expression in the curly brackets is bounded. Next,

$$B_2(x) \le \frac{c}{\sqrt{nx}}\sum_{k=0}^{\infty}\exp\left[-\frac{(k-nx)^2}{2nx}\right] \le c\int_0^{\infty}\exp(-y^2)\mathrm{d}y \le c,$$

[1] In fact, the quoted theorem holds only for $|k-nx| \le n^{\alpha}$, $\alpha < 2/3$. However, it is easy to see from the proof of that theorem that (3.4) still remains true (only the asymptotic equivalence fails to hold).

and

$$B_3(x) \le \frac{c}{\sqrt{x}n^{3/4}} \sum_{k=0}^{\infty} |nx-k| \exp\left[-\frac{(k-nx)^2}{2nx}\right] \le \frac{c\sqrt{x}}{n^{1/4}} \int_0^{\infty} y\exp(-y^2)\mathrm{d}y \le c.$$

Case 2: $x_0 + \frac{10}{n^{3/4}} < x \le \frac{4}{\sqrt{n}}$. Then using

$$\begin{aligned}(nx-k)^2 &\ge \frac{1}{2}(nx - 3\sqrt{n} - n^{1/4})^2 + \frac{1}{2}(3\sqrt{n} + n^{1/4} - k)^2 \\ &\ge \frac{1}{2}n(x\sqrt{n}-3)^2 - n^{3/4}(x\sqrt{n}-3) + \frac{1}{2}(3\sqrt{n} + n^{1/4} - k)^2 \\ &\ge 4n^{3/4}(x\sqrt{n}-3) + \frac{1}{2}(3\sqrt{n} + n^{1/4} - k)^2\end{aligned}$$

as well as (3.1) and (3.4) we obtain

$$\begin{aligned}A_1(x) &\le \frac{c}{n^{3/4}\sqrt{x}} \exp\left[\frac{2}{\sqrt{3x}n^{1/4}}(x\sqrt{n}-3) - \frac{2}{xn^{1/4}}(x\sqrt{n}-3)\right] \\ &\quad \times \sum_{0\le k\le 3\sqrt{n}+n^{1/4}} (3\sqrt{n}+n^{1/4}-k)\exp\left[-\frac{(3\sqrt{n}+n^{1/4}-k)^2}{4nx}\right] \\ &\le \frac{c}{n^{3/4}\sqrt{x}} \sum_{j=1}^{\infty} j\exp\left(-\frac{j^2}{4nx}\right) \le cn^{1/4}\sqrt{x}\int_0^{\infty} y\mathrm{e}^{-y^2}\mathrm{d}y \le c.\end{aligned}$$

Case 3: $\frac{4}{\sqrt{n}} < x \le \frac{1}{2}$. Then by the well-known estimate in connection with Bernstein polynomials

$$p_{n,k}(x) \le 2\exp\left[-\frac{n}{4x}\left(x-\frac{k}{n}\right)^2\right], \quad 0 \le x \le 1, 0 \le k \le 4nx \tag{3.5}$$

(which is obtained from formula (22) of Lorentz [5], p. 19, by choosing $u = \frac{1}{2x}\left|x-\frac{k}{n}\right|$), we have $p_{n,k}(x) \le 2\mathrm{e}^{-c\sqrt{n}}$ while $w(x_0) \le \mathrm{e}^{-n^{1/4}}$; thus $A_1(x) \le \mathrm{e}^{-c\sqrt{n}}$.

Estimate of $A_2(x)$. Again, this sum is non-empty only if $x > 3/\sqrt{n}$. Case 1: $x_0 < x \le 1/n^{1/3}$. Since now $0 < nx - k < n^{2/3}$, we can use (3.4), as well as $x - \frac{k}{n} \ge x^{3/2}$ to obtain by the second inequality of Lemma 1

$$\begin{aligned}A_2(x) &\le \frac{c}{\sqrt{nx}} \sum_{3\sqrt{n}<k\le nx(1-\sqrt{x})} \exp\left[-\frac{n}{2x}\left(x-\frac{k}{n}\right)^2 + \frac{2\sqrt{n}}{x\sqrt{k}}\left(x-\frac{k}{n}\right)\right] \\ &\le \frac{c}{\sqrt{nx}} \sum_{3\sqrt{n}<k\le nx(1-\sqrt{x})} \exp\left\{-\left(x-\frac{k}{n}\right)\left(\frac{n\sqrt{x}}{2} - \frac{2}{x\sqrt{x_0}}\right)\right\} \\ &\le \frac{c}{\sqrt{nx}} \sum_{3\sqrt{n}<k\le nx(1-\sqrt{x})} \exp\left\{-\left(\frac{1}{2}-\frac{2}{9}\right)\sqrt{x}(nx-k)\right\} \\ &\le \frac{c}{\sqrt{nx}} \sum_{j=1}^{\infty} \exp(-c\sqrt{x}j) \le \frac{c}{\sqrt{nx}} \int_0^{\infty} \mathrm{e}^{-y}\mathrm{d}y \le c.\end{aligned} \tag{3.6}$$

Case 2: $1/n^{1/3} < x \leq 1/2$. Then again the same estimate but without the factor $1/\sqrt{nx}$ leads to

$$A_2(x) \leq \sum_{j \geq nx^{3/2}} \mathrm{e}^{-cj\sqrt{x}} \leq \frac{c}{\sqrt{x}} \int_{nx^2}^{\infty} \mathrm{e}^{-cy} \mathrm{d}y \leq \frac{c}{\sqrt{x}} \mathrm{e}^{-cnx^2} \leq n^{1/6} \exp(-cn^{1/3}) \leq c.$$

Estimate of $A_3(x)$. Here $\frac{w(x)}{w(k/n)} \leq c$. This is trivial if $k \geq nx$, and in the opposite case because of $k > (1 - \frac{1}{\sqrt{2}})nx$ and $x - \frac{k}{n} \leq x^{3/2}$ the second inequality in Lemma 1 yields

$$\frac{w(x)}{w(k/n)} \leq \exp\left(\frac{2x^{3/2}}{x\sqrt{k/n}}\right) \leq c,$$

whence $A_3(x) \leq c \sum_{k=0}^{n} p_{n,k}(x) = c$.

Estimate of $A_4(x)$. Now

$$\frac{n^{3/4}}{w(x_3)} \leq \mathrm{e}^{cn^{1/4}},$$

but since $0 \leq x \leq 1/2$ and $k \geq 3n/4$, we evidently have $p_{n,k}(x) \leq \mathrm{e}^{-cn}$ which leads to an exponentially small estimate for $A_4(x)$. Collecting these estimates, we obtain (2.1).

4. Proof of Theorem 1, Part (2.2)

In order to prove the statement we need several lemmas.

Lemma 2. *We have*

$$w(x) \int_x^{1/2} \frac{y-x}{w(y)} \mathrm{d}y \leq cx^3, \quad 0 < x \leq \frac{1}{2}.$$

Of course, an analogous inequality holds for $\frac{1}{2} \leq x < 1$.

Proof. Using the first inequality in Lemma 1 we obtain

$$w(x) \int_x^{1-x} \frac{y-x}{w(y)} \mathrm{d}y \leq \sum_{0 \leq k \leq \frac{1}{\sqrt{x}}-1} \int_{x+kx^{3/2}}^{x+(k+1)x^{3/2}} (k+1)x^{3/2} \exp\left(-\frac{kx^{3/2}}{8x^{3/2}}\right) \mathrm{d}y$$
$$+ \int_{2x}^{1-x} \exp\left(-\frac{x}{8x\sqrt{x}}\right) \mathrm{d}y \leq x^3 \sum_{k=0}^{\infty} k\mathrm{e}^{-k/8} + \mathrm{e}^{-1/(8\sqrt{x})} \leq cx^3$$

(the second integral appears only if $x < 1/3$). □

Lemma 3. *If* $f \in W_2$ *then with the notation* (1.5) *we have*

$$\|(f - F_n(f))w\| \leq c \frac{\|f''\varphi^2 w\|}{n}.$$

Proof. By the definition of $F_n(f)$ it follows that

$$\|(f - F_n(f))w\| \leq \|(f - P_1)w\|_{[0,x_1]} + \|(f - P_2)w\|_{[x_2,1]}.$$

By symmetry, it is sufficient to estimate the first term. Since $f''w \in C_w$, we have

$$f(x) = t_1(x) + \int_x^{x_1} (t-x) f''(t)\mathrm{d}t, \quad t_1 \in \mathcal{P}_1$$

and

$$f(x) - P_1(x) = \int_x^{x_1} f''(t)(t-x)\mathrm{d}t - \frac{x-x_1}{x_0-x_1}\int_{x_0}^{x_1} f''(t)(t-x_0)\mathrm{d}t,$$

whence by Lemma 2

$$\begin{aligned}
&|f(x) - P_1(x)|w(x) \\
&\le w(x)\int_x^{x_1} |f''(t)|(t-x)\mathrm{d}t + n^{3/4}(x_1-x)w(x)\int_{x_0}^{x_1} |f''(t)|(t-x_0)\mathrm{d}t \\
&\le c\|f''\varphi^2 w\| w(x)\left[\frac{1}{x}\int_x^{1/2}\frac{t-x}{w(t)}\mathrm{d}t + \frac{n^{3/4}(x_1-x)}{x_0}\int_{x_0}^{1/2}\frac{t-x_0}{w(t)}\mathrm{d}t\right] \\
&\le c\|f''\varphi^2 w\|\left[x^2 + \frac{n^{5/4}(x_1-x)w(x)}{w(x_0)}x_0^3\right] \le c\|f''\varphi^2 w\|\left[\frac{1}{n} + \frac{(x_1-x)w(x)}{n^{1/4}w(x_0)}\right]. \quad (4.1)
\end{aligned}$$

Thus, it remains to estimate $\frac{(x_1-x)w(x)}{w(x_0)}$. By the second inequality of Lemma 1, $w(x_1) \le cw(x_0)$ and thus

$$\begin{aligned}
\frac{(x_1-x)w(x)}{w(x_0)} &\le c\frac{(x_1-x)w(x)}{w(x_1)} \le c(x_1-x)\exp\left(-\frac{x_1-x}{4x_1\sqrt{x}}\right) \\
&\le c(x_1-x)\mathrm{e}^{-cn^{3/4}(x_1-x)} \le \frac{c}{n^{3/4}},
\end{aligned}$$

which completely proves Lemma 3. □

Lemma 4. *If* $f \in W_2$ *then*

$$\|(F_n - B_n(F_n))w\| \le \frac{c}{n}\|F_n''\varphi^2 w\|. \quad (4.2)$$

Proof. Again, it is sufficient to estimate here for $0 \le x \le 1/2$. Using the fact that F_n'' vanishes on $[0, x_0]$, we obtain

$$\begin{aligned}
&|F_n(x) - B_n(F_n, x)|w(x) = \left|\sum_{k=0}^{n} w(x)p_{n,k}(x)\int_{k/n}^{x}\left(t-\frac{k}{n}\right)F_n''(t)dt\right| \\
&\le c\|F_n''\varphi^2 w\|\left\{\sum_{0\le k\le 3\sqrt{n}} + \sum_{3\sqrt{n}<k\le nx(1-\sqrt{x})} + \sum_{nx(1-\sqrt{x})<k\le 3n/4} + \sum_{3n/4<k\le n}\right\} \quad (4.3) \\
&\left|\int_{\max(k/n,3/\sqrt{n})}^{\max(x,3/\sqrt{n})}\frac{w(x)}{\varphi^2(t)w(t)}\left|t-\frac{k}{n}\right|\mathrm{d}t\right| p_{n,k}(x) =: \|F_n''\varphi^2 w\|\sum_{i=1}^{4} E_i(x).
\end{aligned}$$

Estimate of $E_1(x)$. We may assume that $3/\sqrt{n} \le x \le 1/2$, since otherwise the contribution of this sum is zero.

Case 1: $x_0 \le x \le x_0 + 16/(3n^{3/4})$. Then by the second inequality of Lemma 1 $w(x)/w(t) \le c$, and we obtain by (3.4)

$$\begin{aligned}E_1(x) &\le c\sqrt{n} \sum_{0\le k\le 3\sqrt{n}} \left(x - \frac{k}{n}\right)^2 p_{n,k}(x) \le \frac{c}{n^2\sqrt{x}} \sum_{k=0}^{n} (nx-k)^2 \exp\left(-\frac{(nx-k)^2}{nx}\right)\\ &\le \frac{c}{n^2\sqrt{x}} \sum_{j=0}^{\infty} (j+1)^2 \exp\left(-\frac{j^2}{nx}\right) \le \frac{cx}{\sqrt{n}} \int_0^\infty y^2 \mathrm{e}^{-y^2} \mathrm{d}y \le \frac{c}{n}. \qquad (4.4)\end{aligned}$$

Case 2: $x_0 + 16/(3n^{3/4}) < x \le 4/\sqrt{n}$. Then using again (3.4) we get

$$\begin{aligned}E_1(x) &\le c\frac{\sqrt{n}w(x)}{w(x_0)} \sum_{0\le k\le 3\sqrt{n}} \left(x - \frac{k}{n}\right)^2 p_{n,k}(x)\\ &\le \frac{c}{\sqrt{x}} \sum_{0\le k\le 3\sqrt{n}} \left(x - \frac{k}{n}\right)^2 \exp\left\{-\frac{n}{4x}\left(x - \frac{k}{n}\right)^2\right\}\\ &\quad \times \exp\left\{-\frac{n}{4x}\left(x - \frac{k}{n}\right)^2 + \frac{2}{x_0\sqrt{x}}(x - x_0)\right\}\\ &\le cn^{1/4} \exp\left[-\frac{n}{4x}(x-x_0)^2 + \frac{2}{x_0\sqrt{x}}(x-x_0)\right]\\ &\quad \times \sum_{0\le k\le 3\sqrt{n}} \left(x - \frac{k}{n}\right)^2 \exp\left\{-\frac{n^{3/2}}{16}\left(x - \frac{k}{n}\right)^2\right\}. \qquad (4.5)\end{aligned}$$

Here on the right hand side the function in the square brackets is negative in the interval considered. Thus

$$\begin{aligned}E_1(x) &\le \frac{c}{n^{7/4}} \sum_{j=1}^{\infty} j^2 \exp\left(-\frac{cj^2}{\sqrt{n}}\right) \le \frac{c}{n^{7/4}} \int_1^\infty u^2 \exp\left(-\frac{cu^2}{\sqrt{n}}\right) \mathrm{d}u\\ &\le \frac{c}{n} \int_1^\infty \sqrt{v}\mathrm{e}^{-v} \mathrm{d}v \le \frac{c}{n}.\end{aligned}$$

Case 3: $4/\sqrt{n} < x \le 1/2$. Then using the second inequality in Lemma 1 and (3.5) we obtain

$$\begin{aligned}\frac{w(x)}{\varphi^2(t)w(t)} p_{n,k}(x) &\le c\sqrt{n} \exp\left\{-(x-x_0)\left[-\frac{2}{x\sqrt{x_0}} + \frac{n}{4x}(x-x_0)\right]\right\}\\ &\le c\sqrt{n} \exp\left[-(x-x_0)\left(\frac{n}{4} - \frac{n^{3/4}}{2\sqrt{3}} - \frac{3\sqrt{n}}{4x}\right)\right]\\ &\le c\sqrt{n} \exp\left[-(x-x_0)\left(\frac{n}{4} - \frac{n^{3/4}}{2\sqrt{3}} - \frac{3n}{16}\right)\right] \le \mathrm{e}^{-c\sqrt{n}},\end{aligned}$$

which ensures that $E_1(x)$ is exponentially small.

Estimate of $E_2(x)$: This sum is non-empty only if $x > x_0$. The estimate goes along the same lines as Cases 1 and 2 of estimating $A_2(x)$. In Case 1, in (3.4) we will have the extra factor

$\sqrt{n}\left(x-\frac{k}{n}\right)^2$ which leads to

$$E_2(x) \le \frac{c}{n^2\sqrt{x}} \sum_{j=1}^{\infty} j^2 e^{-cj\sqrt{x}} \le \frac{c}{n^2x^2} \int_1^{\infty} \le \frac{c}{n}.$$

In Case 2, we have used (3.3) instead of (3.2) (i.e. with the factor $1/\sqrt{nx}$ missing) to get

$$E_2(x) \le \frac{c}{n^{3/2}} \sum_{j=0}^{\infty} j^2 e^{-cj\sqrt{x}} \le \frac{c}{(nx)^{3/2}} \int_0^{\infty} y^2 e^{-y} dy \le \frac{c}{n}.$$

Estimate of $E_3(x)$: Here by the second inequality in Lemma 1

$$\frac{w(x)}{w(t)} \le \frac{w(x)}{w(x(1-\sqrt{x}))} \le \exp \frac{2x^{3/2}}{x\sqrt{x}} \le c,$$

and thus

$$E_3(x) \le \frac{c}{x} \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 p_{n,k}(x) \le \frac{c}{n}$$

(cf. Lorentz [5], pp. 5–6).

Estimate of $E_4(x)$: Here the integral over $[x, k/n]$ can be estimated by the integral over the interval $[x, x_3]$, since F_n'' vanishes in $[x_3, 1]$. Then

$$\frac{w(x)}{\varphi^2(t)w(t)} \le \frac{c\sqrt{n}}{w(x_3)} \le e^{cn^{1/4}}, \tag{4.6}$$

but since $0 \le x \le 1/2$ and $k \ge 3n/4$, we evidently have $p_{n,k}(x) \le e^{-cn}$ which leads to an exponentially small estimate for $E_4(x)$. $\square$

Lemma 5. *If* $f \in W_2$ *then*

$$\|F_n''\varphi^2 w\| \le c\|f''\varphi^2 w\|.$$

Proof. By (1.5) it follows that

$$F_n''(f,x) = \begin{cases} 0 & \text{if } 0 \le x \le x_0, \text{ or } x_3 \le x \le 1, \\ [(1-\psi_0(x))P_1(x) + \psi_0(x)f(x)]'' & \text{if } x_0 \le x \le x_1, \\ f''(x) & \text{if } x_1 \le x \le x_2, \\ [(1-\psi_2(x))f(x) + \psi_2(x)P_2(x)]'' & \text{if } x_2 \le x \le x_3, \end{cases}$$

Thus we may assume that $x \in [x_0, x_1]$. We obtain

$$\begin{aligned} F_n''(f,x) &= [\psi_0(x)(f(x) - P_1(x))]'' \\ &= \frac{\psi''\left(\frac{x-x_0}{x_1-x_0}\right)}{(x_1-x_0)^2}(f(x)-P_1(x)) + \frac{2\psi'\left(\frac{x-x_0}{x_1-x_0}\right)}{x_1-x_0}(f(x)-P_1(x))' \\ &\quad + \psi\left(\frac{x-x_0}{x_1-x_0}\right) f''(x). \end{aligned}$$

It follows from the proof of Lemma 3 that

$$\|(f-P_1)\varphi^2 w\|_{[x_0,x_1]} \le c\frac{\|(f-P_1)w\|}{\sqrt{n}} \le c\frac{\|f''\varphi^2 w\|}{n^{3/2}},$$

whence we get

$$\|F_n''\varphi^2 w\|_{[x_0,x_1]} \le C\|f''\varphi^2 w\| + cn^{3/4}\|(f-P_1)'\varphi^2 w\|_{[x_0,x_1]}.$$

For the second term we use the Ditzian inequality [2, p. 15] and $w(x_1) \sim w(x_0)$ to have

$$\begin{aligned}\|(f-P_1)'\varphi^2 w\|_{[x_0,x_1]} &\le c\left\{\frac{1}{x_1-x_0}\|(f-P_1)\varphi^2 w\|_{[x_0,x_1]} + (x_1-x_0)\|f''\varphi^2 w\|\right\}\\ &\le \frac{c}{n^{3/4}}\|f''\varphi^2 w\|_{[x_0,x_1]}\end{aligned}$$

whence the lemma follows. Now (2.2) follows from the partition

$$f - B_n^\star(f) = (f - F_n) + (F_n - B_n(F_n))$$

and from Lemma 3 to 5. $\square$

5. Proof of Theorem 1, Part (2.3)

First we prove Propositions 1 and 2.

Proof of Proposition 1. Let, e.g. $B < C$. Then $\tilde{K}(C, f, t^2)_w \le \tilde{K}(B, f, t^2)_w$. On the other hand, with $Bh = Ch_1$ $\left(h_1 = \frac{B}{C}h\right)$, we have

$$\begin{aligned}\tilde{K}(B, f, t^2)_w &= \sup_{h\le t}\inf\{\|(f-g)w\|_{I(Bh)} + h^2\|g''\varphi w\|_{I(Bh)};\, g\in W^2\}\\ &= \sup_{h\le t}\inf\{\|(f-g)w\|_{I(Ch_1)} + \left(\frac{C}{B}\right)^2 h_1^2\|g''\varphi w\|_{I(Ch_1)};\, g\in W^2\}\\ &\le \left(\frac{C}{B}\right)^2\sup_{h\le t}\inf\{\|(f-g)w\|_{I(Ch_1)} + h_1^2\|g''\varphi w\|_{I(Ch_1)};\, g\in W^2\}\\ &\le \left(\frac{C}{B}\right)^2\tilde{K}\left(C, f, \left(\frac{B}{C}t\right)^2\right) \le \left(\frac{C}{B}\right)^2\tilde{K}(C, f, t^2)_w \quad (B/C<1). \quad \square\end{aligned}$$

Proof of Proposition 2. Observe that for $x \in [2Ch, 1-2Ch]$ we have

$$\eta h \le x - h\varphi(x) < x + h\varphi(x) \le 1 - \eta h \tag{5.1}$$

with $\eta = \left(2C - \frac{1}{4}\right)$. Then, since $w(x) \sim w(y)$ for $x, y \in I(Ch)$ and $|x-y| \le h\varphi(x)$, we have

$$\begin{aligned}\Omega_\varphi^2(C, f, t)_w &= \sup_{h\le t}\|w\Delta_{h\varphi}^2 f\|_{I(Ch)}\\ &\le c\sup_{h\le t}\inf\{\|(f-g)w\|_{I(\eta h)} + h^2\|g''\varphi w\|_{I(\eta h)};\, g\in W^2\}\\ &= c\tilde{K}(\eta, f, t^2)_w \sim \tilde{K}(C, f, t^2)_w\end{aligned}$$

by Proposition 1.

Now let $A > 1$ be a large but fixed real number, $M := \left[\frac{1}{9Ah}\right]$, and

$$t_k := \sin^2\frac{k+\sqrt{M}}{M+2\sqrt{M}}\frac{\pi}{2} \quad \left(h < \frac{1}{9A}, k = 0, \dots, M\right).$$

Then it is easily seen that $t_{k+1} - t_k \sim h\varphi(t_k)$, and $w(x) \sim w(y)$, $x, y \in [t_k, t_{k+1}]$, $k = 0, \ldots, M-1$. With $\Psi \in C^\infty$ a non-decreasing function such that

$$\Psi(x) = \begin{cases} 1, & x \geq 1 \\ 0, & x \leq 0, \end{cases}$$

let

$$\Psi_k(x) = \Psi\left(\frac{x - y_k}{y_{k+1} - y_k}\right),$$

where $y_k = \frac{t_k + t_{k+1}}{2}$. Moreover, let

$$f_\tau(x) = 4 \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \sum_{l=1}^{2} (-1)^{l+1} \binom{2}{l} f(x + l\tau(u_1 + u_2)) \mathrm{d}u_1 \mathrm{d}u_2$$

be the second Steklov function and let us define the functions

$$F_{hk}(x) = \frac{2}{h} \int_{\frac{h}{2}}^{h} f_{\tau\varphi(t_k)}(x) \mathrm{d}\tau \quad (k = 0, \ldots, M),$$

and

$$G_h(x) = \sum_{k=0}^{M} F_{hk}(x) \Psi_{k-1}(x)(1 - \Psi_k(x)), \qquad \Psi_{-1}(x) = \Psi_M(x) = 0.$$

Then, by virtue of (5.1), for some $A > C$, following an argument analogous to that in [4] or [1], it is possible to prove the inequalities

$$\sup_{0<h\leq t} \|(G_h - f)w\|_{I(Ah)} \leq D \sup_{0<h\leq t} \|w \Delta_{h\varphi}^2 f\|_{I(Ch)}$$

and

$$\sup_{0<h\leq t} \|G_h'' \varphi^2 w\|_{I(Ah)} \leq D \sup_{0<h\leq t} \|w \Delta_{h\varphi}^2 f\|_{I(Ch)},$$

with D independent of f and t. Then, using Proposition 1, we deduce

$$\tilde{K}(C, f, t^2)_w \sim \tilde{K}(A, f, t^2)_w \leq D \Omega_\varphi^2(C, f, t)_w$$

and the proof is complete. $\square$

Now we prove

Lemma 6. *For all $g \in W^2$ we have*

$$\inf_{q \in \mathcal{P}_1} \|(g - q)w\|_{[0,2t]} + \inf_{q \in \mathcal{P}_1} \|(g - q)w\|_{[1-2t,1]} \leq ct^2 \|g'' \varphi^2 w\|.$$

Proof. It is sufficient to prove the inequality for the first term on the left hand side. Let $T \in \mathcal{P}_1$ be the linear Taylor polynomial of g at $2t$; then by Lemma 2

$$\begin{aligned} \inf_{q \in \mathcal{P}_1} \|(g - q)w\|_{[0,t]} &\leq \|(g - T)w\|_{[0,t]} \\ &= \left\| w(x) \int_x^{At} (u - x) g''(u) \mathrm{d}u \right\|_{[0,2t]} \leq \|g''\varphi^2 w\| \left\| \frac{w(x)}{x} \int_x^t \frac{u - x}{w(u)} \mathrm{d}u \right\|_{[0,t]} \\ &\leq ct^2 \|g''\varphi^2 w\|. \quad \square \end{aligned}$$

Lemma 7. *For all $f \in C_w$ we have*

$$\omega_\varphi^2(f,t)_w \sim K(f,t^2)_w, \tag{5.2}$$

where the constants involved in "$\sim$" are independent of t and f.

Proof. In order to prove (5.2), we note that

$$\Omega_\varphi^2(f,t)_w \le c\tilde{K}(f,t^2)_w \le cK(f,t^2)_w,$$
$$\inf_{q\in\mathcal{P}_1} \|(f-q)w\|_{[0,2t]} \le \|(f-g)w\|_{[0,2t]} + \inf_{q\in\mathcal{P}_1} \|(g-q)w\|_{[0,2t]}$$

and

$$\inf_{q\in\mathcal{P}_1} \|(f-q)w\|_{[1-2t,1]} \le \|(f-g)w\|_{[1-2t,1]} + \inf_{q\in\mathcal{P}_1} \|(g-q)w\|_{[1-2t,1]}.$$

Then, using Lemma 6 and recalling the definition of ω_φ^2, the upper estimate in (5.2) follows. To prove the inverse inequality in (5.2), let $p_{1,t}, p_{2,t} \in \mathcal{P}_1$ be the linear functions realizing the infimums in (1.6) for a given t. Also, recall that the function G_h in the proof of Proposition 2 satisfies

$$\|(f-G_h)w\|_{[2t,1-2t]} + t^2\|G_h''\varphi^2 w\|_{[2t,1-2t]} \le c\Omega_\varphi^2(f,t)_w \le c\omega_\varphi^2(f,t)_w.$$

Now let

$$z_1 = \frac{t}{2}, \qquad z_2 = t, \qquad z_3 = 1-t, \qquad z_4 = 1-\frac{t}{2}$$

and introduce the function

$$\Gamma_t(x) = \left[1-\psi\left(\frac{x-z_1}{z_2-z_1}\right)\right]p_{1,t}(x) + \psi\left(\frac{x-z_1}{z_2-z_1}\right)\left[1-\psi\left(\frac{x-z_2}{z_3-z_2}\right)\right]G_h(x) + \psi\left(\frac{x-z_2}{z_3-z_2}\right)p_{2,t}(x)$$

that "glues together" $p_{1,t}$, $p_{2,t}$ and G_h. Following a usual procedure (see for instance [4]), it is routine work to prove the estimates

$$K(f,t^2)_w \le c\|(\Gamma_t - f)w\| + t^2\|\Gamma_t''\varphi^2 w_\varphi\| \le c\omega_\varphi^2(f,t)_w$$

which concludes the proof of (5.2). $\square$

Now we are ready to prove (2.3). By (2.1), (2.2) and (5.2), for any $f \in C_w$ and $g \in W^2$ realizing the infimum in (5.1) with $t = 1/\sqrt{n}$ we get

$$\begin{aligned}\|(f-B_n^\star(f))w\| &\le \|w(f-g)\| + \|B_n^\star(f-g)w\| + \|(g-B_n^\star(g))w\| \\ &\le c\left[\|(f-g)w\| + \frac{1}{n}\|g''\varphi^2 w\|\right] \le cK\left(f,\frac{1}{n}\right)_w \le c\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right)_w.\end{aligned}$$

6. Proof of Theorem 2

By (2.3), it is sufficient to prove that

$$\|w(f-B_n^\star(f))\| = O(n^{-\alpha/2}) \Rightarrow \omega_\varphi^2(f,h)_w = O(h^\alpha), \quad 0<\alpha<2. \tag{6.1}$$

Lemma 8. *We have*

$$\|F_n w\| \le c\|fw\|.$$

Proof. By (1.5) and the boundedness of the ψ function, it suffices to prove

$$\|P_1 w\|_{[0,x_1]} + \|P_2 w\|_{[x_2,1]} \le c\|fw\|. \tag{6.2}$$

Using the representation (1.2) we obtain

$$\begin{aligned}|P_1(x)|w(x) &\le cn^{3/4}\|fw\|w(x)\left(\frac{x_1-x}{w(x_0)}+\frac{|x-x_0|}{w(x_1)}\right)\\ &\le cn^{3/4}\|fw\|\left(\frac{(x_1-x)w(x)}{w(x_1)}+\frac{|x-x_0|w(x)}{w(x_0)}\right), \quad 0<x\le x_1,\end{aligned}$$

since by Lemma 1, $w(x_0)\sim w(x_1)$. Now using Lemma 1 again,

$$\frac{(x_1-x)w(x)}{w(x_1)} \le (x_1-x)\mathrm{e}^{-cn^{3/4}(x_1-x)} \le \frac{c}{n^{3/4}}$$

and similarly

$$\frac{|x-x_0|w(x)}{w(x_0)} \le \frac{c}{n^{3/4}}.$$

This settles the estimate of the first term in (6.2); the second term can be estimated analogously. □

Lemma 9. *We have for* $\lambda = 1, 2$,

$$w(x)\sum_{k=0}^{n}\left|F_n\left(\frac{k}{n}\right)\right|\cdot\left|x-\frac{k}{n}\right|^{\lambda} p_{n,k}(x)$$
$$\le \begin{cases} c\sqrt{w(x)}\|fw\|, & \text{if } 0<x\le \frac{1}{n} \text{ or } 1-\frac{1}{n}\le x<1,\\ c\dfrac{\varphi^{\lambda}(x)}{n^{\lambda/2}}\|fw\|, & \text{if } \frac{1}{n}<x<1-\frac{1}{n}.\end{cases}$$

Proof. Let first $\frac{1}{n}\le x\le 1-\frac{1}{n}$. By the Cauchy–Schwarz inequality we obtain

$$w(x)\sum_{k=0}^{n}\left|F_n\left(\frac{k}{n}\right)\right|\cdot\left|x-\frac{k}{n}\right|^{\lambda} p_{n,k}(x)$$
$$\le \left[w^2(x)\sum_{k=0}^{n}F_n^2\left(\frac{k}{n}\right)p_{n,k}(x)\cdot\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^{2\lambda}p_{n,k}(x)\right]^{1/2} := \sqrt{B_1B_2}.$$

Here, applying the proof of (2.1) for w^2 and f^2 instead of w and f, respectively,

$$B_1 \le \|f^2w^2\| = \|fw\|^2,$$

while

$$B_2 \le c\left(\frac{\varphi^{2\lambda}(x)}{n^{\lambda}}+\frac{\varphi^{2(\lambda-1)}(x)}{n^{\lambda+1}}\right) \le c\frac{\varphi^{2\lambda}(x)}{n^{\lambda}}, \quad \frac{1}{n}\le x\le 1-\frac{1}{n}, \lambda=1,2$$

(cf. Lorentz [5], p. 14), and this proves the statement in the interval specified. Now if e.g. $0 < x \le \frac{1}{n}$, then by (3.2)–(3.3)

$$w(x)\sum_{k=0}^{n}\left|F_n\left(\frac{k}{n}\right)\right|\cdot\left|x-\frac{k}{n}\right|^{\lambda} p_{n,k}(x) \le c\frac{n^{5/4}w(x)}{w(1/\sqrt{n})}\|fw\|$$
$$\le c\|fw\|n^{5/4}\exp\left(-\frac{1}{\sqrt{x}}+n^{1/4}\right) \le \mathrm{e}^{-1/(2\sqrt{x})}\|fw\|, \quad 0 < x \le 1/n. \quad \square$$

Lemma 10. *We have*

$$\|B_n^{\star''}(f)\varphi^2 w\| \le cn\|fw\|, \quad f \in C_w.$$

Proof. We consider the following representation:

$$B_n^{\star''}(f,x)\varphi^2(x) = -nB_n(F_n,x) + \frac{n^2}{\varphi^2(x)}\sum_{k=0}^{n}F_n\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)^2 p_{n,k}(x)$$
$$-\frac{n(1-2x)}{\varphi^2(x)}\sum_{k=0}^{n}F_n\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)p_{n,k}(x),$$

whence by Lemma 9,

$$|B_n^{\star''}(f,x)|\varphi^2(x)w(x) \le cnB_n^{\star}(f,x)w(x) + \frac{n^2}{\varphi^2(x)}w(x)\sum_{k=0}^{n}\left|F_n\left(\frac{k}{n}\right)\right|\left(x-\frac{k}{n}\right)^2 p_{n,k}(x)$$
$$+\frac{n}{\varphi^2(x)}w(x)\sum_{k=0}^{n}\left|F_n\left(\frac{k}{n}\right)\right|\left|x-\frac{k}{n}\right|p_{n,k}(x) \le c\left(n+\frac{\sqrt{n}}{\varphi(x)}\right)\|fw\| \le cn\|fw\|. \quad \square$$

Lemma 11. *We have*

$$\|B_n^{\star''}(f)\varphi^2 w\| \le c\|f''\varphi^2 w\|, \quad f \in W_2.$$

Proof. We consider the representation

$$B_n^{\star''}(f,x) = n(n-1)\sum_{k=0}^{n-2}\overrightarrow{\Delta}_{1/n}^{2} f\left(\frac{k}{n}\right)p_{n-2,k}(x)$$

(see e.g. [2], formula (34)), whence

$$|B_n^{\star''}(f,x)|\varphi^2(x)w(x) \le c\|f''\varphi^2 w\|\sum_{k=1}^{n-3}\frac{w(x)\varphi^2(x)}{w(k/n)\varphi^2(k/n)}p_{n-2,k}(x).$$

Here the sum is of the same character as in the proof of (2.1), but with $w\varphi^2$ instead of w. It is easy to see that all the considerations performed in proving (2.1) remain in effect, and the sum proves to be bounded. $\square$

After these preparations, the proof of (6.1) is completely analogous to that used in the proof of (9) in [2], applying Lemma 9.3.4 from [4,3]. We omit the details.

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